Math 206B Lecture 27 Notes

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1 Operations on D-Finite Series

Note: Today's lecture is a guest lecture.

1.1 Addition, multiplication, and composition of D-finite series

Last time, we showed that $u \in K[\![x]\!]$ is D-finite iff $\dim_{K(x)}(\mathrm{span}(\{u, u', u'', \dots\})) < \infty$.

Theorem 1.1. *The set D of D-finite $u \in K[\![x]\!]$ is a subalgebra of $K[\![x]\!]$. If $u, v \in D$ and $\alpha, \beta \in K$, then $\alpha u + \beta v \in D$, and $uv \in D$.*

Proof. Given $w \in K[\![x]\!]$, let $V_w = \mathrm{span}_{K(x)}(\{w, w', w'', \dots\}) \subseteq K((x))$. Suppose $u, v \in D$, $\alpha, \beta \in K$, and let $y = \alpha u + \beta v$. Then $y, y', y'', \dots \in V_u + V_v$. Thus, $\dim(V_y) \leq \dim(V_u + V_v) \leq \dim(V_u) + \dim(V_v) < \infty$.

Next, let $u, v \in D$. Consider $\phi : V_u \otimes_{K(x)} V_r \to K((x))$ defined by $\phi(y \otimes z) = yz$ for all $y \in V_u$ and $z \in V_v$. The product rule implies $V_{uv} \subseteq \phi(V_u \otimes_{K(x)} V_v)$; indeed, $(uv)^{(i)} = \sum_{j=0}^{i} \binom{i}{j} u^{(i)} v^{(i-j)}$. Thus, $\dim(V_{uv}) \leq \dim(V_u \otimes_{K(x)} V_v) = \dim(V_u)\dim(V_v) < \infty$, so $uv \in D$. $\square$

Theorem 1.2. *Let $u \in D$ and $v \in K_{\mathrm{alg}}[\![x]\!]$ (i.e. $v \in K[\![x]\!]$ and v is algebraic over $K(x)$) with $v(0) = 0$. Then $u(v(x)) \in D$.*

Proof. Let $y = u(v(x))$. Then $y' = u'(v(x))v'(x)$, $(u'(v(x)))' = u''(v(x))v'(x)$, etc. In general, $y^{(i)}$ is a linear combination of $u(v(x)), u'(v(x)), u''(v(x)), \dots$ with coefficients in $K[v, v', v'', \dots]$. Since v is algebraic over $K(x)$, $v^{(i)} \in K(x, v)$ for all i (proved last time). Thus, $K[v, v', \dots] \subseteq K(x, v)$.

Let $V = \mathrm{span}_{K(x)}(\{u(v(x)), u'(v(x)), \dots\}) \ni y^{(i)}$ for all i. We want to show that $\dim_{K(x)}(V) < \infty$. Since u is D-finite, $\dim_{K(x)}(\mathrm{span}_{K(x)}(\{u(x), u'(x), \dots\})) < \infty$. By "specializing x at v," $\dim_{K(x)}(\mathrm{span}_{K(x)}(\{u(v(x)), u'(v(x)), \dots\})) < \infty$. So we get that $\dim_{K(x,v)}(\mathrm{span}_{K(x,v)}(\{u(v(x)), u'(v(x)), \dots\})) < \infty$. Then $\dim_{K(x,v)}(V) < \infty$, and (since v is algebraic over $K(x)$) $[K(x, v) : K(x)] < \infty$, so

$$\dim_{K(x)}(V) = (\dim_{K(x,v)}) \cdot [K(x, v) : K(x)] < \infty. \qquad \square$$

Example 1.1. We know that $\sum_{n\geq 0} n!x^n$, e^x, and $\frac{x}{\sqrt{1-4x}}$ are in D. So we can get that $u = (\sum_{n\geq 0} n!x^n)e^{x/\sqrt{1-4x}} \in D$. This would be difficult to do by hand without the results we have proved.

1.2 Hadamard products of P-recursive series

Given $h : \mathbb{N} \to K$ and $R(n) \in K(n)$, we want to define $Rh : \mathbb{N} \to K$ by $Rh(n) = R(n)h(n)$. But this could be undefined when $R(n) = \infty$. Here is the solution: given $h_1, h_2 : \mathbb{N} \to K$, say $h_1 \sim h+2$ if $h_1(u) = h_2(u)$ for all $n \gg 0$. Call $[h]$, the equivalence class of h, the **germ** of h. Define $\mathcal{G} = \{[h] \mid h : \mathbb{N} \to K\}$; this is a $\mathbb{K}(n)$ vector space. Note: if $g \sim h$, the nh is P-recursive iff g is P-recursive . For each $h : \mathbb{N} \to K$, set $\mathcal{G}_h = \mathrm{span}_{K(n)}(\{[h(n)], [h(n+1)], [h(n+2)], \dots\})$.

Fact: h is P-recursive iff $\dim_{K(n)} \mathcal{G}_h < \infty$.

Definition 1.1. Given $u = \sum_n f(n)x^n$ and $v = \sum_n g(n)x^n$, define the **Hadamard product** $u * v = \sum_{n\geq 0} f(n)g(n)x^n$.

Theorem 1.3. *If $f, g : \mathbb{N} \to K$ are P-recursive, then so is fg. That is, $u, v \in D \implies u * v \in D$.*

Proof. It is sufficient to show that if $[f], [g]$ are P-recursive (element member of the germ is recursive), then so is $[fg]$. Define $\phi : \mathcal{G}_f \otimes_{K(n)} \mathcal{G}_g \to \mathcal{G}$ such that for each i, j, on simple tensors, $\phi([f(n+i)] \otimes [g(n+j)]) = [f(n+i)][g(n+j)] = [f(n+i)g(n+j)]$. So the image of ϕ contains $\mathcal{G}_{fg} = \mathrm{span}_{K(n)}(\{[f(n)g(n)], [f(n+1)g(n+1)], \dots\})$. So

$$\dim_{K(n)}(\mathcal{G}_{fg}) \leq \dim_{K(n)}(\mathcal{G}_f \otimes \mathcal{G}_g) = (\dim(\mathcal{G}_f))(\dim(\mathcal{G}_g)) < \infty.$$

So fg is P-recursive. □

1.3 Fun facts

Here are some fun facts from Professor Pak's 2016 206A notes:

Theorem 1.4. *Let $S \subseteq \mathbb{Z}^d$ with $|S| < \infty$. Let a_n be the number of walks $0 \to 0$ of length n on $\mathbb{Z}^d$ with steps in S. Then (a_n) is P-recursive.*

Theorem 1.5. *If $a_n = |\{\sigma \in S_n : \sigma^2 = 1\}|$, then $a_n = a_{n-1} + (n-1)a_{n-2}$, so (a_n) is P-recursive.*

Definition 1.2. Given $F = \sum f(n_1, n_2, \dots, n_r)x_1^{n_1} \cdots x_r^{n_r} \in F[\![a_1, \dots, x_r]\!]$, define the **diagonal**, $\mathrm{diag}(F) \in K[\![t]\!]$, by

$$(\mathrm{diag}(F))(t) = \sum_n f(n, n, \dots, n)t^n.$$

Theorem 1.6 (Furstenberg)**.** *Suppose* $F(s,t) \in K[\![s,t]\!] \cap K(s,t)$*. Then* $\operatorname{diag}(F)$ *is algebraic. If* $P, Q \in \mathbb{Z}[x_1, \ldots, x_r]$*, then* $\operatorname{diag}(P/Q)$ *is D-finite.*

The proof of this theorem involves Puiseaux series.

Remark 1.1. The converse is also true. An algebraic single variable power series is the diagonal of such a multivariable series.